ON NECESSARY AND SUFFICIENT CONDITIONS FOR L²-WELL-POSEDNESS OF MIXED PROBLEMS FOR HYPERBOLIC EQUATIONS

By

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1. Introduction.

Let $\boldsymbol{R}_+^n$ be the open half space $\{(x, y); x>0, y \in \boldsymbol{R}^{n-1}\}$. We consider the mixed problems $(P, B_j; j=1, \cdots, l)$, denote simply it by (P, B_j), for hyperbolic equations of order m:

$$
\begin{aligned}
&(P(\partial_t, D_x, D_y)\, u)(t, x, y) = f(t, x, y) && \text{in } (0, T)\times \boldsymbol{R}_+^n,\\
&(B_j(\partial_t, D_x, D_y)\, u)(t, 0, y) = 0 \qquad (j=1, \cdots, l) && \text{in } (0, T)\times \boldsymbol{R}^{n-1},\\
&(\partial_t^k u)(0, x, y) = 0 \qquad (k=0, 1, \cdots, m-1) && \text{in } \boldsymbol{R}_+^n,
\end{aligned}
$$

where $\partial_t = \dfrac{\partial}{\partial t}$, $D_x = -i\dfrac{\partial}{\partial x}$, $D_y = \left(-i\dfrac{\partial}{\partial y_1}, \cdots, -i\dfrac{\partial}{\partial y_{n-1}}\right)$ and $i=\sqrt{-1}$.

The purpose of this paper is to determine the necessary and sufficient conditions for well-posedness in the following sense:

Definition. *The mixed problem (P, B_j) is L^2-well-posed with decreasing order ν (ν; non negative integer) if and only if there exist positive constants C, T and T' with $T' \leq T$ which satisfy the following condition:*

For every $f \in H^{\nu+1,0}((-\infty, T)\times \boldsymbol{R}_+^n)$[1)] with $f=0$ $(t<0)$ the mixed problem (P, B_j) has a unique solution $u \in H^m((0, T')\times \boldsymbol{R}_+^n)$ such that

$$
(1.\ 1)\qquad \int_0^{T'} |||u(t, \cdot, \cdot)|||_{m-1}^2\, dt \leq C\int_0^T |||f(t, \cdot, \cdot)|||_{\nu,0}^2\, dt^{2)}.
$$

When $\nu=0$ we call it *L^2-well-posedness* (with decreasing order 0).

The contents of this paper are as follows. In Section 2 we give a summary on boundary value problems for elliptic ordinary differential equations depending on parameters. In Section 3 we investigate the zeros of Lopatinskii's determinant under the L^2-well-posedness with decreasing order ν. In

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1), 2) For the definitions see Section 2. Hereafter we denote various positive constants by C and C', which are independent of variables in each inequality considered below respectively.

Section 4 we describe a certain necessary and sufficient condition for L^2-well-posedness with decreasing order ν by the term of the compensating function, and moreover we describe it by the terms of the reflection coefficients in Section 5. In T. Shirota and K. Asano [9] it has been shown by semi-group method that the mixed problem (P, D_x^{2j-1}) $(m=2l)$ is well posed in the stronger sense if $P(D)=P(\partial_t, D_x, D_y)$ does not contain the odd order terms in D_x. As one of the applications of the results in Section 5 we show that, in the case of constant coefficients, the property of $P(D)$ mentioned above is essential to be L^2-well-posed (with decreasing order 0) for the mixed problem (P, D_x^{2j-1}). This assertion is achieved in Section 6. Finally we present some examples in Section 7.

This paper contains the details of our previous paper [8].

2. Preliminaries.

First of all we state fundamental assumptions. Let $P(D)$ and $B_j(D)$ be *homogeneous* differential operators of order m and $m_j (m_j<m)$ with *constant* coefficients respectively. We assume that $P(D)$ is *strictly hyperbolic with respect to t-direction* and *the hyperplane* $x=0$ *is non-characteristic for* $P(D)$. Then it is easily seen that the number $l(m-l)$ of the roots $\lambda_j^+(\tau, \sigma)$ $(\lambda_j^-(\tau, \sigma))$, located in the upper (lower) half λ-plane, in λ of the characteristic equation $P(\tau, \lambda, \sigma)=0$ is constant for any $(\tau, \sigma)\in \boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}$ respectively, where $\boldsymbol{C}_+ = \{\tau\in \boldsymbol{C}\,;\ \boldsymbol{Re}\,\tau>0\}$. Furthermore we assume that *the hyperplane* $x=0$ *is non-characteristic for* $B_j(D)$ *and* $m_j>m_k$ *if* $j>k$.

Throughout this paper we use the following Fourier-Laplace transforms and norms:

$$\hat{u}(\tau, \lambda, \sigma)=\int_0^\infty dt\int_0^\infty dx\int_{\boldsymbol{R}^{n-1}} e^{-\tau t-i\lambda x-i\sigma y}u(t, x, y)\,dy\,,$$

$$\hat{u}(\tau, x, \sigma)=\int_0^\infty dt\int_{\boldsymbol{R}^{n-1}} e^{-\tau t-i\sigma y}u(t, x, y)\,dy\,,$$

$$|||u(t, \cdot, \cdot)|||_k^2=\sum_{j=0}^{k}\|(\partial_t^j u)(t, \cdot, \cdot)\|_{k-j}^2\,,$$

$$|||u(t, \cdot, \cdot)|||_{k,l}^2=\sum_{j=0}^{l}\sum_{h+j+|\alpha|=0}^{k}\int_{\boldsymbol{R}^{n-1}}dy\int_0^\infty |(D_x^j\partial_t^h D_y^\alpha u)(t, x, y)|^2\,dx\,,$$

$$\left.\begin{aligned} |||\hat{u}(\tau, \cdot, \cdot)|||_k^2&=\sum_{j=0}^{k}\int_{\boldsymbol{R}^{n-1}}(|\tau|^2+|\sigma|^2)^{k-j}d\sigma\int_0^\infty |(D_x^j\hat{u})(\tau, x, \sigma)|^2\,dx\,,\\ |||\hat{u}(\tau, \cdot, \cdot)|||_{k,l}^2&=\sum_{j=0}^{l}\sum_{h=0}^{k-j}\int_{\boldsymbol{R}^{n-1}}(|\tau|^2+|\sigma|^2)^{k-h}\,d\sigma\int_0^\infty |(D_x^j\hat{u})(\tau, x, \sigma)|^2\,dx\,,\end{aligned}\right\}\quad(\tau\in\boldsymbol{C}_+)$$

where $\sigma y=\sigma_1 y_1+\cdots+\sigma_{n-1}y_{n-1}$, $|\sigma|^2=\sigma_1^2+\cdots+\sigma_{n-1}^2$ and $\|\cdot\|_j$ is the norm in

Sobolev space $H^j(\boldsymbol{R}_+^n)$ (j; a non negative integer). By $H^{k,l}((-\infty, T)\times\boldsymbol{R}_+^n)$ (k, l; non negative integer) we understand the completion of $C_0^\infty(\overline{(-\infty, T)\times\boldsymbol{R}_+^n})$ by the norm $\left(\int_{-\infty}^T |||u(t,\cdot,\cdot)|||_{k,l}^2\, dt\right)^{\frac{1}{2}}$.

We define *Lopatinskii's determinant* $R(\tau, \sigma)$ as follows:

$$B(\tau, \sigma) = \det\left(B_1(\tau, \lambda_j^+(\tau, \sigma), \sigma), \cdots, B_l(\tau, \lambda_j^+(\tau, \sigma), \sigma)\,;\, j\downarrow 1, \cdots, l\right),$$
$$R(\tau, \sigma) = B(\tau, \sigma) \Big/ \prod_{1\le j<k\le l} (\lambda_j^+(\tau, \sigma) - \lambda_k^+(\tau, \sigma)).$$

Then $R(\tau, \sigma)$ is analytic in $\boldsymbol{C}_+\times\boldsymbol{R}^{n-1}$ and can be continuously extended to $\overline{\boldsymbol{C}}_+\times\boldsymbol{R}^{n-1}$. Let V be the zeros of $R(\tau, \sigma)$ in $\boldsymbol{C}_+\times\boldsymbol{R}^{n-1}$ and for every $\tau\in\boldsymbol{C}_+$ let $S(\tau)$ be the analytic variety $\{\sigma\in\boldsymbol{R}^{n-1}\,;\,(\tau, \sigma)\in V\}$ in $\boldsymbol{R}^{n-1}$. Then we have $aV=V$ and $aS(\tau)=S(a\tau)$ for every $a>0$.

Applying now the Fourier-Laplace transform to equations in the mixed problem (P, B_j) we obtain the boundary value problem $(\hat{P}, \hat{B}_j)$ for elliptic ordinary differential equations depending on parameters $(\tau, \sigma)\in\boldsymbol{C}_+\times\boldsymbol{R}^{n-1}$:

$$(P(\tau, D_x, \sigma)\,\hat{u})(\tau, x, \sigma) = \hat{f}(\tau, x, \sigma) \qquad \text{in } \boldsymbol{R}_+^1,$$
$$(B_j(\tau, D_x, \sigma)\,\hat{u})(\tau, 0, \sigma) = 0 \qquad (j=1, \cdots, l).$$

Let $R_j(\tau, x, \sigma)$ be the determinat replacing j-column in $R(\tau, \sigma)$ by the transposed vector of $(\mathbf{exp}(ix\lambda_1^+(\tau, \sigma)), \cdots, \mathbf{exp}(ix\lambda_l^+(\tau, \sigma))$ and $\Gamma=\Gamma(\tau, \sigma)$ a closed Jordan curve in the lower half λ-plane enclosing all the roots $\lambda_k^-(\tau, \sigma)$ $(k=1, \cdots, m-l)$. If $R(\tau, \sigma)\neq 0$ for some $(\tau, \sigma)\in\boldsymbol{C}_+\times\boldsymbol{R}^{n-1}$, it is well known that for every $\hat{f}(\tau, \cdot, \sigma)\in C_0^\infty(\boldsymbol{R}_+^1)$ the boundary value problem $(\hat{P}, \hat{B}_j)$ has a unique solution $\hat{u}(\tau, \cdot, \sigma)\in C^\infty(\overline{\boldsymbol{R}_+^1})$, which is written in the following form:

$$\hat{u}(\tau, x, \sigma) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{i\lambda x}\hat{f}(\tau, \lambda, \sigma)}{P(\tau, \lambda, \sigma)}\, d\lambda + \frac{1}{2\pi i}\int_0^\infty G(x, s, \tau, \sigma)\,\hat{f}(\tau, s, \sigma)\, ds, \tag{2. 1}$$

where $G(x, s, \tau, \sigma) = -\sum_{j=1}^l \frac{R_j(\tau, x, \sigma)}{R(\tau, \sigma)}\int_\Gamma \frac{B_j(\tau, \lambda, \sigma)}{P(\tau, \lambda, \sigma)}\, e^{-is\lambda}\, d\lambda$.

3. Zeros of Lopatinskii's determinant.

In this section we investigate the zeros of Lopatinskii's determinant $R(\tau, \sigma)$ under L^2-well-posedness with decreasing order ν. The following theorem shows that if the mixed problem (P, B_j) is L^2-well-posed with decreasing order ν then the zeros V of $R(\tau, \sigma)$ has the product representation $\boldsymbol{C}_+\times S$, where S is the cone surface with vertex at the origin in $\boldsymbol{R}^{n-1}$.

Theorem 3. 1. *If the mixed problem (P, B_j) is L^2-well-posed with decreasing order ν, then the analytic varieties $S(\tau)$ don't depend on $\tau\in\boldsymbol{C}_+$.*

Proof. It suffices to prove that if $R(\tau, \sigma_0)$ is identically not zero for some $\sigma_0 \in \boldsymbol{R}^{n-1}$ then $R(\tau, \sigma_0) \neq 0$ for any $\tau \in \boldsymbol{C}_+$.

Using Weierstrass preperation theorem, if $R(\tau_0, \sigma_0)=0$ for some $\tau_0 \in \boldsymbol{C}_+$ then there are a neighbourhood $U(\sigma_0)$ in $\boldsymbol{C}^{n-1}$ and a continuous (analytic) function $\tau(\sigma)$ in an open set $D \subset U(\sigma_0)$ such that $R(\tau(\sigma), \sigma)=0$ and $\tau(\sigma) \in \boldsymbol{C}_+$ in D. Hence we can find continuous functions $a_j(\sigma)\,(j=1, \cdots, l)$ in D such that for any $\sigma \in D$

$$(3.1) \qquad (a_1(\sigma), \cdots, a_l(\sigma)) \neq 0\,,$$

$$(3.2) \qquad \sum_{j=1}^{l} a_j(\sigma)\,\beta_{h,j}(\tau(\sigma), \sigma)=0 \qquad (h=1, \cdots, l)\,,$$

where $\beta_{h,1}(\tau, \sigma)=B_h(\tau, \lambda_1^+(\tau, \sigma), \sigma)\,,$

$$\beta_{h,j}(\tau, \sigma)=\int_0^1 d\theta_1 \cdots d\theta_{j-2}\int_0^1 \theta_1^{j-2}\cdots\theta_{j-2}(D_\lambda^{j-1}B_h)(\tau, g_j(\tau, \sigma\,;\,\theta), \sigma)\,d\theta_{j-1}\,,$$

$$g_j(\tau, \sigma\,;\,\theta)=\lambda_1^+(\tau, \sigma)+(\lambda_2^+(\tau, \sigma)-\lambda_1^+(\tau, \sigma))\,\theta_1+\cdots$$
$$\cdots+(\lambda_j^+(\tau, \sigma)-\lambda_{j-1}^+(\tau, \sigma))\,\theta_1\cdots\theta_{j-1} \qquad (j=2, \cdots, l)\,.$$

Note that we can determine branches λ_j^+ such that $\lambda_j^+(\tau(\sigma), \sigma)$ is continuous (analytic) in $D' \subset D$, because the hyperplane $x=0$ is non-characteristic for $P(D)$.

First we construct smooth solutions of the equations $(P(D)\,u)(t, x, y)=0$ and $(B_j(D)\,u)(t, 0, y)=0$, which don't satisfy the following estimate

$$(3.3) \qquad \int_0^T |||u(t, \cdot, \cdot)|||_{m-1}^2\,dt \leq C\,\|(\partial_t^k u)(0, \cdot, \cdot)\|_h^2,$$

where non negative integers h and k are arbitrarily fixed.

For this purpose we define the function

$$u(t, x, y)=\sum_{j=1}^{l}\int_{D''} a_j(\sigma)\,\gamma_j(\tau(\sigma), x, \sigma)\,\mathbf{exp}(\tau(\sigma)t+i\sigma y)\,d\sigma,$$

where $D''=D' \cap \boldsymbol{R}^{n-1}$ which may be assumed to be not empty, $\gamma_1(\tau, x, \sigma)=\mathbf{exp}(ix\lambda_1^+(\tau, \sigma))$ and

$$\gamma_j(\tau, x, \sigma)=(ix)^{j-1}\int_0^1 d\theta_1\cdots d\theta_{j-2}\int_0^1 \theta_1^{j-2}\cdots\theta_{j-2}\,\mathbf{exp}(ixg_j(\tau, \sigma\,;\theta))\,d\theta_{j-1} \qquad (j \geq 2).$$

For every positive integer p let us set $u_p(\mathrm{t}, x, y)=u(pt, px, py)$. Then, by the homogeneity of $P(D)$ and $B^j(D)$ and (3.2), u_p is a solution of the equations above. From Plancherel theorem we obtain

$$(3.4) \qquad \|(\partial_t^k u_p)(0, \cdot, \cdot)\|_h^2 \leq C\sum_{j=0}^{h} p^{2k+2j-n}\,.$$

On the other hand, since for each h $(D_x^h\gamma_j)(\tau(\sigma), x, \sigma)$ $(j=1, \cdots, l)$ are linearly independent as functions in x, it follows from (3. 1) that $\sum_{j=1}^{l} a_j(\sigma)(D_x^h\gamma_j)(\tau(\sigma), x, \sigma)\not\equiv 0$ for any $\sigma\in D''$. Hence choosing sufficiently small D'' we have, by Plancherel theorem, for any positive T and p

$$\int_0^T |||u_p(t, \cdot, \cdot)|||_{m-1}^2 dt \geq C' e^{CTp} \sum_{k=1}^{m-1} p^{2k-n}. \tag{3. 5}$$

By (3. 4) and (3. 5) u_p does not satisfy the estimate (3. 3) for a sufficiently large p.

Next we construct a solution of the mixed problem (P, B_j) which does not satisfy the estimate (1. 1). Using the above function u_p let us define for a large K

$$v_p(t, x, y) = \sum_{k=0}^{K} t^k (\partial_t^k u_p)(0, x, y)/k!. \tag{3. 6}$$

Setting $w'_p(t, x, y) = \sum_{q=0}^{L} x^q f_{p,q}(t, y)$ $(L = \max\ m_j)$, we determine the $f_{p,q}$ such that w'_p is a solution of the following equations:

$$\begin{aligned} &(B_j(D)(w'_p - v_p))(t, 0, y) = 0 \qquad (j = 1, \cdots, l) \qquad \text{in } (0, T)\times \boldsymbol{R}^{n-1}, \\ &(\partial_t^k(w'_p - v_p + u_p))(0, x, y) = 0 \qquad (k=0, 1, \cdots, m-1) \quad \text{in } \boldsymbol{R}_+^n. \end{aligned} \tag{3. 7}$$

This is possible if $K>m+L$. In fact, substituting w'_p into the first equation in (3. 7), the $f_{p,q}$ are inductively determined by the following forms:

$$\begin{aligned} &f_{p,q}(t, y) = 0 \text{ if } q\neq m_j \qquad (j = 1, \cdots, l), \\ &f_{p,m_1}(t, y) = (B_1(D) v_p)(t, 0, y)/m_1!, \\ &f_{p,m_j}(t, y) = \Big\{-\sum_{h=1}^{j-1} m_h! A^j_{m_j-m_h}(\partial_t, D_y) f_{p,m_h}(t, y) + (B_j(D) v_p)(t, 0, y)\Big\}\Big/ m_j! \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j\geq 2), \end{aligned} \tag{3. 8}$$

where $B_j(D) = D_x^{m_j} + \sum_{d=0}^{m_j} A_d^j(\partial_t, D_y)\, D_x^{m_j-d}$. Note that w'_p satiefies then the second equation in (3. 7). Let us set $w_p(t, x, y) = \varphi(x)\, w'_p(t, x, y)$, where $\varphi(x) \in C_0^\infty(\overline{R_+^1})$ with $\varphi(x) = 1\left(0\leq x\leq \frac{1}{2}\right)$ and $\varphi(x) = 0\ (x\geq 1)$. Then $w_p - v_p + u_p \in H^m((0, T)\times \boldsymbol{R}_+^n)$ is a solution of the mixed problem (P, B_j) (setting $f = P(D)(w_p - v_p + u_p)$) and satisfies $P(D)(w_p - v_p + u_p)(0, x, y) = 0$ in $\boldsymbol{R}_+^n$.

If the mixed problem (P, B_j) is L^2-well-posed with decreasing order ν, then a solution $w_p - v_p + u_p$ must satisfy the estimate (1. 1), that is,

$$\int_0^{T'} |||(w_p - v_p + u_p)(t, \cdot, \cdot)|||_{m-1}^2 dt \leq C \int_0^T |||(P(D)(w_p - v_p + u_p))(t, \cdot, \cdot)|||_{\nu,0}^2 dt.$$

Using $P(D)\, u_p = 0$ we obtain

$$(3.9)\qquad \int_0^{T'} |||u_p(t,\cdot,\cdot)|||_{m-1}^2 dt \leq C\left(\int_0^T |||w_p(t,\cdot,\cdot)|||_{m+\nu}^2 dt + \int_0^T |||v_p(t,\cdot,\cdot)|||_{m+\nu}^2 dt\right).$$

Since we have from (3. 8)

$$|||w_p(t,\cdot,\cdot)|||_{m+\nu}^2 \leq C|||v_p(t,\cdot,\cdot)|||_{m+L+1+\nu}^2,$$

it follows from (3. 6) and (3. 9) that

$$(3.10)\qquad \int_0^{T'} |||u_p(t,\cdot,\cdot)|||_{m-1}^2 dt \leq C\sum_{k=0}^{K} ||(\partial_t^k u_p)(0,\cdot,\cdot)||_{m+L+1+\nu}^2.$$

The right hand in (3.10) is polynomial order in p and the left hand in (3.10) is exponential order in p. Then, for a sufficiently large p, u_p is a required function. Thus the proof is complete.

Remark. The construction of the function $w_p - v_p$ is used in Section 4.

Corollary 3.2. *Suppose that $B_j(D)$ don't contain the terms in ∂_t. If the mixed problem (P, B_j) is L^2-well-posed with decreasing order ν, then $S(\tau)$ is empty for any $\tau \in \boldsymbol{C}_+$.*

Proof. Assume that $R(\tau_0, \sigma_0)=0$ for some $(\tau_0, \sigma_0)\in \boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}$. By Theorem 3. 1 and the homogeneity of $R(\tau, \sigma)$ we obtain that $R(\tau, 0)=0$ for any $\tau\in \boldsymbol{C}_+$. On the other hand, $R(\tau, 0)$ is written in the following form:

$$R(\tau, 0) = \frac{\det((ia_k\tau)^{m_1}, \cdots, (ia_k\tau)^{m_l}\,;\ k\downarrow 1, \cdots, l)}{\det((ia_k\tau)^0, \cdots, (ia_k\tau)^{l-1}\,;\ k\downarrow 1, \cdots, l)}$$

where the $ia_k\tau(\tau\in \boldsymbol{C}_+)$ are roots of $P(\tau, \lambda, 0)=0$ with positive imaginary part. By the theory of characters of unitary groups[3] weo btain that $R(\tau, 0)\neq 0$ for every $\tau\neq 0$, here we use the assumption that $m_1<\cdots<m_l$. (see [10]). Thus the proof is complete.

By the first part of the proof of Corollary 3. 2 we have

Corollary 3.3. *Suppose that $R(\tau_0, 0)\neq 0$ for some $\tau_0 \in \boldsymbol{C}_+$. If the mixed problem (P, B_j) is L^2-well-posed with decreasing order ν, then V is empty.*

4. A certain necessary and sufficient condition for L^2-well-posedness (I).

In this section we describe a certain necessary and sufficient condition for L^2-well-posedness with decreasing order ν by the term of the compensating function $G(x, s, \tau, \sigma)$.

Let Σ_+ be the set $\{(\tau', \sigma')\in \boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}\,;\ |\tau'|^2+|\sigma'|^2=1\}$ and $\bar{\Sigma}_+$ its closure.

3) This method is suggested by Prof. L. Gårding.

Let $V' = V \cap \Sigma_+$, where V is the zeros of $R(\tau, \sigma)$ in $\boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}$. When A is a subset in Σ_+ we denote the complement of A in Σ_+ by A^c.

Then we obtain the following main

Theorem 4.1. *Let $S(\tau) \neq \boldsymbol{R}^{n-1}$ and $S(\tau) = S$ be independent of $\tau \in \boldsymbol{C}_+$. The mixed problem (P, B_j) is L^2-well-posed with decreasing order ν if and only if the following condition (I) is satisfied*:

(I). For every $(\tau_0', \sigma_0') \in (\bar{\Sigma}_+ - \Sigma_+) \cup V'$ there exist a neighbourhood $U(\tau_0', \sigma_0')$ and a constant $C(\tau_0', \sigma_0')$ such that for any $(\tau', \sigma') \in U(\tau_0', \sigma_0') \cap \Sigma_+ \cap V'^c$

$$\|(D_x^k G)(x, s, \tau', \sigma')\|_{\mathfrak{L}(L^2(s>0), L^2(x>0))} \leq C(\tau_0', \sigma_0')(\boldsymbol{Re}\,\tau')^{-\nu-1} \quad (k = 0, 1, \cdots, m-1) \tag{4.1}$$

where $\|\cdot\|_{\mathfrak{L}(L^2(s>0), L^2(x>0))}$ is the operator norm from $L^2(s>0)$ to $L^2(x>0)$.

To prove Theorem 4.1 we need the following lemmas. In particular, Lemma 4.3 is essential. First we state an elementary lemma on strictly hyperbolic polynomial without proof.

Lemma 4.2. *If a homogeneous polynomial $P(\tau, \xi)$ of degree m is strictly hyperbolic with respect to τ, then we have for any $(\tau, \xi) \in \boldsymbol{C}_+ \times \boldsymbol{R}^n$*

$$|P(\tau, \xi)|^2 \geq C(\boldsymbol{Re}\,\tau)^2 (|\tau|^2 + |\xi|^2)^{m-1}.$$

Lemma 4.3. *If the assumption in Theorem 4.1 and the condition (I) are satisfied, then for every $(\tau, \sigma) \in \boldsymbol{C}_+ \times \boldsymbol{R}^{n-1}$ with $\sigma \notin S$ and $f \in H^{\nu+h+1, h}((-\infty, \infty) \times \boldsymbol{R}_+^n)$ (h; a non negative integer) with $f = 0$ ($t < 0$) the boundary value problem $(\hat{P}, \hat{B}_j)$ has a unique solution $\hat{u}(\tau, \cdot, \sigma) \in H^{m+h}(\boldsymbol{R}_+^1)$, which satisfies the following estimate*

$$\begin{aligned} &(\boldsymbol{Re}\,\tau)^{2(\nu+1)} \|\hat{u}(\tau, \cdot, \cdot)\|_{m-1}^2 \leq C \|\hat{f}(\tau, \cdot, \cdot)\|_{\nu,0}^2, \\ &(\boldsymbol{Re}\,\tau)^{2(\nu+1)} \|\hat{u}(\tau, \cdot, \cdot)\|_{m+k}^2 \leq C \|\hat{f}(\tau, \cdot, \cdot)\|_{\nu+k+1,k}^2 \quad (k = 0, 1, \cdots, h) \end{aligned} \tag{4.2}$$

for any $\tau \in \boldsymbol{C}_+$

Proof. Note that the notation of the norms has no ambiguity, because *the measure of S is zero.* Using the limit process and the relation $\hat{P}(\tau, D_x, \sigma)\hat{u}(\tau, x, \sigma) = \hat{f}(\tau, x, \sigma)$, it suffices to prove the first estimate of (4.2) for $f \in C_0^\infty((0, \infty) \times \boldsymbol{R}_+^n)$ and $\hat{u}$ defined in (2.1).

First let $\hat{u}_1$ be the first term in the right hand of (2.1). Then it follows from Lemma 4.2 and Plancherel theorem that

$$(\boldsymbol{Re}\,\tau)^2 \|\hat{u}_1(\tau, \cdot, \cdot)\|_{m-1}^2 \leq C \|\hat{f}(\tau, \cdot, \cdot)\|_0^2,$$

which implies the first estimate of (4.2) for $\hat{u}_1$.

Let U' be the set of all points (τ', σ') for which (4.1) is valid and U the set $\{(\tau, \sigma); (\tau', \sigma') \in U'\}$ where $\tau' = \tau\rho^{-1}$, $\sigma' = \sigma\rho^{-1}$ and $\rho = (|\tau|^2 + |\sigma|^2)^{\frac{1}{2}}$. We decompose $\hat{u}=\hat{u}_1+\hat{u}_2+\hat{u}_3$, where $\hat{u}_3$ is the function multiplying the characteristic function of U to the second term in the right hand of (2. 1).

Next to prove the first estimate of (4. 2) for $\hat{u}_2$ we remark the following facts. By the homogeneity of $P(D)$ and $B_j(D)$ we have, for every integer $k\geq 0$ and $(\tau, \sigma)\in U$,

$$\int_0^\infty\left|\frac{(D_x^k R_j)(\tau, x, \sigma)}{R(\tau, \sigma)}\right|^2 dx=\rho^{2k-2m_j-1}\int_0^\infty\left|\frac{(D_x^k R_j)(\tau', x, \sigma')}{R(\tau', \sigma')}\right|^2 dx. \tag{4. 3}$$

If $K=K(\tau', \sigma')$ is the convex hull of the roots $\lambda_j^+(\tau', \sigma')(j=1, \cdots, l)$, then we have

$$\begin{aligned}&|(D_x^k R_j)(\tau', x, \sigma')|\\ &\quad\leq\left\{\sum_{h=0}^{l-1}\sup_{\lambda\in K}|D_\lambda^h(\lambda^k e^{ix\lambda})|/h!\right\}\prod_{p\neq j}\left\{\sum_{h=0}^{l-1}\sup_{\lambda\in K}|(D_\lambda^h B_p)(\tau', \lambda, \sigma')/h!\right\}.\end{aligned} \tag{4. 4}$$

By the fact that $R(\tau', \sigma')\geq C$ and $\boldsymbol{Im}\,\lambda_j^+(\tau', \sigma')\geq C$ for ony $(\tau', \sigma')\in U'$, it follows from (4. 3) and (4. 4) that

$$\int_0^\infty\left|\frac{(D_x^k R_j)(\tau, x, \sigma)}{R(\tau, \sigma)}\right|^2 dx\leq C\rho^{2k-2m_j-1}\qquad\text{for any } (\tau, \sigma)\in U. \tag{4. 5}$$

Since $|P(\tau, \lambda, \sigma)|^2\geq C(|\tau|^2+|\lambda|^2+|\sigma|^2)^m$ for any $(\tau, \sigma)\in U$ and $\lambda\in\boldsymbol{R}^1$, we have

$$\int_{-\infty}^\infty\left|\frac{B_j(\tau, \lambda, \sigma)}{P(\tau, \lambda, \sigma)}\right|^2 d\lambda\leq C\rho^{2m_j-2m+1}\qquad\text{for any } (\tau, \sigma)\in U. \tag{4. 6}$$

By Plancherel theorem and Schwarz inequality it follows from (4. 5) and (4. 6) that

$$|||\hat{u}_2(\tau, \cdot, \cdot)|||_m^2\leq C|||\hat{f}(\tau, \cdot, \cdot)|||_0^2,$$

which implies the first estimate of (4. 2) for $\hat{u}_2$.

Finally note that $(\bar{\Sigma}_+-\Sigma_+)\cup V'$ is compact and

$$(D_x^k G)(x, s, \tau, \sigma)=\rho^{k-m+1}(D_x^k G)(\rho x, \rho s, \tau', \sigma')\qquad(k=0, 1, \cdots, m-1). \tag{4. 7}$$

Using the relation (4. 7), the condition (I) and the change of variables we obtain

$$\begin{aligned}|||\hat{u}_3(\tau, \cdot, \cdot)|||_{m-1}^2&=\sum_{k=0}^{m-1}\int_{D(\tau)}\rho^{2(m-1-k)}d\sigma\int_0^\infty dx\left|\int_0^\infty(D_x^k G)(x, s, \tau, \sigma)\,\hat{f}(\tau, s, \sigma)\,ds\right|^2\\ &=\sum_{k=0}^{m-1}\int_{D(\tau)}\rho^{-3}d\sigma\int_0^\infty dx\left|\int_0^\infty(D_x^k G)(x, s, \tau', \sigma')\,\hat{f}(\tau, s\rho^{-1}, \sigma)\,ds\right|^2\\ &\leq C(\boldsymbol{Re}\,\tau)^{-2(\nu+1)}\int_{D(\tau)}\rho^{2\nu-1}d\sigma\int_0^\infty|\hat{f}(\tau, s\rho^{-1}, \sigma)|^2 ds\\ &\leq C(\boldsymbol{Re}\,\tau)^{-2(\nu+1)}|||\hat{f}(\tau, \cdot, \cdot)|||_{\nu,0}^2,\end{aligned}$$

where $D(\tau)=\{\sigma\in\boldsymbol{R}^{n-1};\ (\tau, \sigma)\in U\}$. Thus the proof is complete.

By Lemma 4.3 and Paley-Wiener theorem (e.g. Theorem 7.1 in [2]) we obtain the following

Lemma 4.4. *If the assumption in Theorem* 4.1 *and the condition* (I) *is satisfied, then for every* $a>0$ *and* f *with* $e^{-at}f\in H^{\nu+h+1,h}((-\infty,\infty)\times \boldsymbol{R}_+^n)$ (h; *a non negative integer*) *and* $f=0$ ($t<0$) *the mixed problem* (P, B_j) (*with* $T=\infty$) *has a unique solution* u *with* $e^{-at}u\in H^{m+h}((0,\infty)\times \boldsymbol{R}_+^n)$ *such that*

$$(4.8)\qquad \begin{aligned} &\int_0^\infty e^{-2at}|||u(t,\cdot,\cdot)|||_{m-1}^2 dt\leq (C/a^{2(\nu+1)})\int_0^\infty e^{-2at}|||f(t,\cdot,\cdot)|||_{\nu,0}^2 dt. \\ &\int_0^\infty e^{-2at}|||u(t,\cdot,\cdot)|||_{m+k}^2 dt\leq (C/a^{2(\nu+1)})\int_0^\infty e^{-2at}|||f(t,\cdot,\cdot)|||_{\nu+k+1,k}^2 dt \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k=0,\ 1,\cdots,h). \end{aligned}$$

The following lemma used in proof of the necessity of Theorem 4.1 is due to T. Okubo. The proof is easy.

Lemma 4.5. *Let* f *be a function in* $H^{\nu,0}((-\infty,\infty)\times \boldsymbol{R}_+^n)$ *with its support in* $(0,T)\times \boldsymbol{R}_+^n$ *and* u *a function satisfying* $e^{-a't}u\in H^m((0,\infty)\times \boldsymbol{R}_+^n)$ *for some* $a'>0$ *and* $(\partial_t^k u)(0,x,y)=0$ ($k=0,1,\cdots,m-1$). *Then the estimate*

$$(4.9)\qquad \int_0^\infty e^{-2a't}|||u(t,\cdot,\cdot)|||_{m-1}^2 dt\leq C\int_0^\infty |||f(t,\cdot,\cdot)|||_{\nu,0}^2 dt$$

implies that

$$(4.10)\qquad |||\hat{u}(\tau,\cdot,\cdot)|||_{m-1}^2\leq C_0 C\int_{-\infty}^{+\infty}|||\hat{f}(a+i\eta,\cdot,\cdot)|||_{\nu,0}^2 d\eta$$

for any τ *with* $\boldsymbol{Re}\,\tau=a$ ($a>a'$), *where a constant* C_0 *depends only on* a, a' *and the support of* f.

Remark. If the mixed problem (P, B_j) is L^2-well-posed with decreasing order ν, then for every $f\in H^{\nu+1,0}((-\infty,\infty)\times \boldsymbol{R}_+^n)$ with $f=0$ ($t<0$) the mixed problem (P, B_j) (with $T=\infty$) has a unique solution u satisfying $e^{-at}u\in H^m((0,\infty)\times \boldsymbol{R}_+^n)$ for some $a>0$ and the estimate (4.9). This follows from the fact that $P(D)$ and $B_j(D)$ are homogeneous and of constant coefficients.

Now we ready for the proof of Theorem 4.1. Our proof of the necessity is inspired by M. Ikawa [4].

Proof of Theorem 4.1. 1). Sufficiency. The existence follows immediately from Lemm 4.4. To prove the uniqueness we use an extension $\tilde{u}$ of u to $(0,\infty)\times \boldsymbol{R}_+^n$ which satisfies $e^{-at}\tilde{u}\in H^m((0,\infty)\times \boldsymbol{R}_+^n)$ for some $a>0$ and $(B_j(D)\tilde{u})(t,0,y)=0$ in $(0,\infty)\times \boldsymbol{R}^{n-1}$. (For the existence of such an extension see Remark after the proof of Theorem 3.1. Here we use the fact that u may be considered as a sufficiently smooth function in (t,y) by the method of convolution). Then the uniqueness follows from Lemma 4.4 and Paley-

Wiener theorem.

2). Necessity. Suppose that the mixed problem (P, B_j) is L^2-well-posed with decreasing order ν. Let f be a smooth function with its support in $(0, T)\times \boldsymbol{R}^n_+$. Then it follows from the Remark above that the estimate (4.10) is valid for the Fourier-Laplace transform $\hat{u}$ of a unique solution u of the mixed problem (P, B_j) (with $T=\infty$). We use notations in the proof of Lemma 4.3. Since the inverse Fourier-Laplace transform u_1 of $\hat{u}_1$ is a solution for Cauchy problem, by Lemma 4.5, the estimate (4.10) is valid for $\hat{u}_1$. Hence $\hat{u}_2+\hat{u}_3$ must also satisfy (4.10), that is,

$$(4.11)\qquad |||(\hat{u}_2+\hat{u}_3)(\tau, \cdot, \cdot)|||^2_{m-1}\leq C\int_{-\infty}^{+\infty}|||\hat{f}(a+i\eta, \cdot, \cdot)|||^2_{\nu,0}d\eta$$

for any τ with $\boldsymbol{Re}\,\tau=a$.

If the condition (I) is not satisfied, then there exist a point $(\tau_0', \sigma_0')\in(\bar{\Sigma}_+ -\Sigma_+)\cup V'$, an integer $k_0(0\leq k_0\leq m-1)$ and a sequence $\{\tau_p', \sigma_p'\}$ $(p=1, 2, 3, \cdots)$ in $\Sigma_+\cap V'^c$ which converges to (τ_0', σ_0') such that

$$(4.12)\qquad C_p=(\boldsymbol{Re}\,\tau_p')^{\nu+1}\|(D_x^{k_0}G)(x, s, \tau_p', \sigma_p')\|_{\mathfrak{L}(L^2(s>0), L^2(x>0))}$$

tends to infinity if p does so.

First we take $g_p, \varphi_p\in C_0^\infty(R^1_+)$ which are identically not zero and satisfy

$$\left|\int_0^\infty\overline{\varphi_p(x)}\,dx\int_0^\infty(D_x^{k_0}G)(x, s, \tau_p', \sigma_p')\,g_p(s)\,ds\right|$$
$$\geq\frac{1}{2}\|\varphi_p\|_{L^2(\boldsymbol{R}^1_+)}\|g_p\|_{L^2(\boldsymbol{R}^1_+)}\|(D_x^{k_0}G)(x, s, \tau_p', \sigma_p')\|_{\mathfrak{L}(L^2(s>0), L^2(x>0))}.$$

Since $\int_0^\infty\overline{\varphi_p(x)}\,dx\int_0^\infty(D_x^{k_0}G)(x, s, \tau', \sigma')\,g_p(s)\,ds$ is continuous at (τ_p', σ_p')[4], there is $\theta_p>0$ such that

$$(4.13)\qquad \begin{aligned}&\left|\int_0^\infty\overline{\varphi_p(x)}\,dx\int_0^\infty(D_x^{k_0}G)(x, s, \tau', \sigma')\,g_p(s)\,ds\right|\\ &\geq\frac{1}{4}\|\varphi_p\|_{L^2(\boldsymbol{R}^1_+)}\|g_p\|_{L^2(\boldsymbol{R}^1_+)}\|(D_x^{k_0}G)(x, s, \tau_p', \sigma_p')\|_{\mathfrak{L}(L^2(s>0), L^2(x>0))}\end{aligned}$$

for any (τ', σ') with $|\sigma'-\sigma_p'|<\theta_p$ and $|\tau'-\tau_p'|<\theta_p$.

Let (τ_p, σ_p) be a point $(a/\boldsymbol{Re}\,\tau_p')(\tau_p', \sigma_p')$ and J_p a set $\{(a/\boldsymbol{Re}\,\tau_p')(\sigma_p'+\delta)\,;\,|\delta|<\alpha_p<\theta_p\}$. Take a sufficiently small α_p such that if $\sigma\in J_p$ then $|\rho_p^{-1}\sigma-\sigma_p'|<\theta_p$, $|\rho_p^{-1}\tau_p-\tau_p'|<\theta_p$ and $C\leq(\boldsymbol{Re}\,\tau_p')\rho_p^{-1}\leq C'$, where $\rho_p=(|\tau_p|^2+|\sigma|^2)^{\frac{1}{2}}$.

Next take $\psi_p\in C_0^\infty(J_p)$ which is identically not zero. We choose $f\in C_0^\infty$

4) Regarding the continuity of the roots $\lambda_j(\tau, \sigma)$ we use the following classical fact: For a fixed point (τ_0, σ_0) there is a labelling of the roots $\lambda_j(\tau, \sigma)$ such that $\lambda_j(\tau, \sigma)$ is continuous at (τ_0, σ_0).

$((0, T))$ such that $\hat{f}_p(\tau_p)=\int_0^\infty e^{-at}f(t)\,dt\neq 0$ if we set $f_p(t)=f(t)\mathbf{exp}(it\,\boldsymbol{Im}\,\tau_p)$. Then we have

$$\int_{-\infty}^{+\infty}|(a+i\eta)^j\hat{f}_p(a+i\eta)|^2d\eta\leqq\sum_{h=0}^{j}|\boldsymbol{Im}\,\tau_p|^{2h}\int_0^\infty e^{-2at}|(\partial_t^{j-h}f)(t)|^2dt.$$

Replacing τ and $\hat{f}(a+i\eta, x, \sigma)$ in (4.11) by τ_p and $\hat{f}_p(a+i\eta)\,g_p(\rho_p x)\,\overline{\psi_p(\sigma)}$ respectively and multiplying $\left(\int_{J_p}d\sigma\int_0^\infty|\psi_p(\sigma)\,\varphi_p(\rho_p x)|^2dx\right)^{\frac{1}{2}}$ to (4.11), it follows from Schwarz inequality that

$$\begin{aligned}&\left(\int_{-\infty}^{+\infty}d\eta\int_{J_p}d\sigma\int_0^\infty\sum_{j=0}^{\nu}(|a+i\eta|^2+|\sigma|^2)^j|\hat{f}_p(a+i\eta)\,g_p(\rho_p x)\,\psi_p(\sigma)|^2dx\right)^{\frac{1}{2}}\\&\qquad\times\left(\int_{J_p}d\sigma\int_0^\infty|\psi_p(\sigma)\,\varphi_p(\rho_p x)|^2dx\right)^{\frac{1}{2}}\\&\geq C\left|\int_{J_p}\psi_p(\sigma)\,\rho_p^{m-1-k_0}d\sigma\int_0^\infty\varphi_p(\rho_p x)\,dx\right.\\&\qquad\left.\times\int_0^\infty(D_x^{k_0}G)\,(x, s, \tau_p, \sigma)\,\hat{f}_p(\tau_p)\,g_p(\rho_p s)\,\overline{\psi_p(\sigma)}\,ds\right|,\end{aligned}$$

which implies by the relation (4.7) and the change of variables that

$$\begin{aligned}&\|g_p\|_{L^2(\boldsymbol{R}_+^1)}\|\varphi_p\|_{L^2(\boldsymbol{R}_+^1)}\left(\sum_{j=0}^{\nu}\int_{-\infty}^{\infty}d\eta\int_{J_p}\rho_p^{-1}(|a+i\eta|^2\right.\\&\qquad\left.+|\sigma|^2)^j|\hat{f}_p(a+i\eta)\,\psi_p(\sigma)|^2d\sigma\right)^{\frac{1}{2}}\left(\int_{J_p}\rho_p^{-1}|\psi_p(\sigma)|^2d\sigma\right)^{\frac{1}{2}}\\&\geq C\left|\hat{f}_p(\tau_p)\right|\left|\int_{J_p}\rho_p^{-2}|\psi_p(\sigma)|^2d\sigma\int_0^\infty\varphi_p(x)\,dx\int_0^\infty(D_x^{k_0}G)\,(x, s, \tau_p\rho_p^{-1}, \sigma\rho_p^{-1})\,g_p(s)\,ds\right|.\end{aligned}$$

From this and the choice of J_p we have

$$\begin{aligned}&\|g_p\|_{L^2(\boldsymbol{R}_+^1)}\|\varphi_p\|_{L^2(\boldsymbol{R}_+^1)}\|\psi_p\|_{L^2(\boldsymbol{R}^{n-1})}^2\left(\sum_{j=0}^{\nu}\int_{-\infty}^{\infty}(|a+i\eta|^2+(\boldsymbol{Re}\,\tau_p')^{-2})^j|\hat{f}_p(a+i\eta)|^2d\eta\right)^{\frac{1}{2}}\\&\geq C(\boldsymbol{Re}\,\tau_p')\left|\int_{J_p}|\psi_p(\sigma)|^2d\sigma\int_0^\infty\varphi_p(x)\,dx\int_0^\infty(D_x^{k_0}G)\,(x, s, \tau_p\rho_p^{-1}, \sigma\rho_p^{-1})\,g_p(s)\,ds\right|.\end{aligned}\tag{4.14}$$

By the choice of f_p and $\boldsymbol{Im}\,\tau_p=(a\,\boldsymbol{Im}\,\tau_p')/\boldsymbol{Re}\,\tau_p'$ the left hand in (4.14) is estimated by

$$\leq C(\boldsymbol{Re}\,\tau_p')^{-\nu}\|g_p\|_{L^2(\boldsymbol{R}_+^1)}\|\varphi_p\|_{L^2(\boldsymbol{R}_+^1)}\|\psi_p\|_{L^2(\boldsymbol{R}^{n-1})}^2.$$

By the choice of g_p and φ_p the integrand in the right hand of (4.14) is not zero at (τ_p', σ_p'). Hence if the diameter of J_p is sufficiently small the right hand of (4.14) is estimated by

$$\geq (C/\sqrt{2})(\boldsymbol{Re}\,\tau_p')\int_{J_p}|\psi_p(\sigma)|^2 d\sigma\left|\int_0^\infty \varphi_p(x)\,dx\int_0^\infty (D_x^{k_0}G)(x, s, \tau_p\rho_p^{-1}, \sigma\rho_p^{-1})\,g_p(s)\,ds\right|,$$

where the constant C is same in (4. 14). Therefore it follows from them, (4. 12) and (4. 13) that $1\geq C\ C_p$. But this inequality is not valid for a sufficiently large p. Thus the proof is complete.

5. A certain necessary and sufficient condition for L^2-well-posedness (II).

In this section we decribe a certain necessary and sufficient condition for L^2-well-posedness with decreasing order ν by the terms of the reflection coefficients. To achieve this purpose we first state the following condition introduced by S. Agmon [1].

Condition (♯). *The multiplicity of a real root $\lambda(\tau, \sigma)$ in λ of the characteristic equation $P(\tau, \lambda, \sigma)=0$ is at most double for every non zero (τ, σ) with $\boldsymbol{Re}\,\tau=0$* and $\sigma\in \boldsymbol{R}^{n-1}$.

To define the reflection coefficients, for every $(\tau_0', \sigma_0')\in(\bar{\Sigma}_+-\Sigma_+)\cup V'$ we arrange the roots $\lambda_j^+(\tau', \sigma')$ into q groups $\{\lambda_{k,h}^+(\tau', \sigma')\,;\ h=1, \cdots, k'\}$ $(k=1, \cdots, q)$ in a sufficiently small neighbourhood $U(\tau_0', \sigma_0')\cap\Sigma_+$ such that $\{\lambda_{k,h}^+(\tau_0', \sigma_0')\,;\ h=1, \cdots, k'\}$ is k'-multiple root. Let $R_{j,k}(\tau', x, \sigma')$ be the determinant replacing the j-colum in $R_j(\tau', x, \sigma')$ by the transposed vector of $(0, \cdots, 0, \mathbf{exp}\,(ix\lambda_{k,1}^+(\tau', \sigma')), \cdots, \mathbf{exp}\,(ix\lambda_{k,k'}^+(\tau', \sigma')), 0, \cdots, 0)$. Since $R_j(\tau', x, \sigma')=\sum_{k=1}^{q}R_{j,k}(\tau', x, \sigma')$, we can define *the generalized reflection coefficients* $C_{k,h}(\tau', \lambda, \sigma')$ $(k=1, \cdots, q\,;\ h=1, \cdots, k')$ by the following equality

$$\sum_{j=1}^{l}\frac{R_j(\tau', x, \sigma')}{R(\tau', \sigma')}B_j(\tau', \lambda, \sigma')=\sum_{k=1}^{q}\sum_{h=1}^{k'}C_{k,h}(\tau', \lambda, \sigma')\,\gamma_{k,h}(\tau', x, \sigma'),\tag{5. 1}$$

where $\gamma_{k,1}(\tau', x, \sigma')=\mathbf{exp}\,(ix\lambda_{k,1}^+(\tau', \sigma'),$

$$\gamma_{k,h}(\tau', x, \sigma')=(ix)^{h-1}\int_0^1 d\theta_1\cdots d\theta_{j-2}\int_0^1\theta_1^{h-2}\cdots\theta_{h-2}\,\mathbf{exp}\,(ixg_{k,h}(\tau', \sigma', \theta))d\theta_{h-1},$$

$$g_{k,h}(\tau', \sigma'\,;\ \theta)=\lambda_{k,1}^+(\tau', \sigma')+(\lambda_{k,2}^+(\tau', \sigma')-\lambda_{k,1}^+(\tau', \sigma'))\,\theta_1+\cdots$$
$$\cdots+(\lambda_{k,h}^+(\tau', \sigma')-\lambda_{k,h-1}^+(\tau', \sigma'))\,\theta_1\cdots\theta_{h-1}\qquad (h\geq 2).$$

In particular, if $\lambda_j^+(\tau', \sigma')$ is simple in $U(\tau_0', \sigma_0')\cap\Sigma_+$, for example $\lambda_j^+(\tau_0', \sigma_0')$ $(\boldsymbol{Re}\,\tau_0'=0)$ is real and the condition (♯) is satisfied[5], then the generalized reflection coefficient is written in the following form:

$$C_j(\tau', \lambda, \sigma')=\widetilde{B}_j(\tau', \lambda, \sigma')/B(\tau', \sigma'),\tag{5. 2}$$

5) See Lemma 6. 1.

where $\widetilde{B}_j(\tau', \lambda, \sigma')$ is the determinant replacing $\lambda_j^+(\tau', \sigma')$ in $B(\tau', \sigma')$ by λ.

From Theorem 4. 1 we obtain the following

Theorem 5. 1. *Suppose that the condition* ($\sharp$) *is satisfied,* $S(\tau) \neq \boldsymbol{R}^{n-1}$ *and* $S = S(\tau)$ *is independent of* τ. *The mixed problem* (P, B_j) *is* L^2*-well-posed with decreasing order* ν *if and only if the following condition* (*II*) *is satisfied*:

(*II*). *For every* $(\tau_0', \sigma_0') \in (\bar{\Sigma}_+ - \Sigma_+) \cup V'$ *there exist a neighbourhood* $U(\tau_0', \sigma_0')$ *and a constant* $C(\tau_0', \sigma_0')$ *such that*

$$(5.3) \qquad \left\| \int_\Gamma \frac{C_{k,h}(\tau', \lambda, \sigma')}{P(\tau', \lambda, \sigma')} e^{-is\lambda} d\lambda \right\|_{L^2(\boldsymbol{R}_+^1)} \leq C(\tau_0', \sigma_0') (\boldsymbol{Im}\, \lambda_{k,h}^+(\tau', \sigma'))^{\frac{1}{2}} (\boldsymbol{Re}\, \tau')^{-\nu-1}$$
$$(k=1, \cdots, q\, ;\ h=1, \cdots, k')$$

for any $(\tau', \sigma') \in U(\tau_0', \sigma_0') \cap \Sigma_+ \cap V'^c$.

Proof. 1). (II) implies (I). By (5.1), (5.3) and the definition of $G(x, s, \tau, \sigma)$ we obtain, for every $f \in L^2(\boldsymbol{R}_+^1)$,

$$(5.4) \qquad \begin{aligned} &\| (D_x^j G)(x, s, \tau', \sigma') f(s)\, ds \|_{L^2(\boldsymbol{R}_+^1)} \\ &\leq \|f\|_{L^2(\boldsymbol{R}_+^1)} C(\tau_0', \sigma_0') (\boldsymbol{Re}\, \tau')^{-\nu-1} \sum_{k=0}^{q} \sum_{h=1}^{k'} \|(D_x^j \gamma_{k,h})(\tau', x, \sigma)\|_{L^2(\boldsymbol{R}_+^1)} (\boldsymbol{Im}\, \lambda_{k,h}^+(\tau', \sigma'))^{\frac{1}{2}} \end{aligned}$$

in a sufficiently small neighbourhood $U(\tau_0', \sigma_0') \cap \Sigma_+ \cap V'^c$.

In $U(\tau_0', \sigma_0') \cap \Sigma_+ \cap V'^c$, if $\boldsymbol{Im}\, \lambda_{k,1}^+(\tau_0', \sigma_0') = 0$, by the condition ($\sharp$), $\lambda_{k,1}^+(\tau', \sigma')$ is simple and hence we have $\|(D_x^j \gamma_{k,1})(\tau', x, \sigma')\|_{L^2(\boldsymbol{R}_+^1)} \leq C\ \ \boldsymbol{Im}\, \lambda_{k,1}^+(\tau', \sigma'))^{-\frac{1}{2}}$, and furthermore if $\boldsymbol{Im}\, \lambda_{k,h}^+(\tau', \sigma') > 0$ we have $\|(D_x^j \gamma_{k,h})(\tau', x, \sigma')\|_{L^2(\boldsymbol{R}_+^1)} \leq C$ and $\boldsymbol{Im}\, \lambda_{k,h}^+(\tau', \sigma') \leq C$. Therefore the condition (I) follows from (5. 4).

2). (I) implies (II). By the condition (I), the definition of $G(x, s, \tau, \sigma)$ and Schwarz inequality we obtain, for every $g, \varphi \in L^2(\boldsymbol{R}_+^1)$,

$$(5.5) \qquad \begin{aligned} &\|g\|_{L^2(\boldsymbol{R}_+^1)} \|\varphi\|_{L^2(\boldsymbol{R}_+^1)} C(\tau_0', \sigma_0') (\boldsymbol{Re}\, \tau')^{-\nu-1} \\ &\geq \left| \sum_{k=1}^{q} \sum_{h=1}^{k'} \left(\int_0^\infty g(s)\, ds \int_\Gamma \frac{C_{k,h}(\tau', \lambda, \sigma')}{P(\tau', \lambda, \sigma')} e^{-is\lambda} d\lambda \right) \left(\int_0^\infty \gamma_{k,h}(\tau', \lambda, \sigma') \varphi(x)\, dx \right) \right| \end{aligned}$$

in a sufficiently small neighbourhood $U(\tau_0', \sigma_0') \cap \Sigma_+ \cap V'^c$.

If the condition (II) is not satisfied, then there exist a point $(\tau_0', \sigma_0') \in (\bar{\Sigma}_+ - \Sigma_+) \cup V'$, a pair of integers (k_0, h_0) $(1 \leq k_0 \leq q\, ;\ 1 \leq h_0 \leq k')$ and a sequence $\{(\tau_p', \sigma_p')\}$ in $\Sigma_+ \cap V'^c$ which converges to (τ_0', σ_0') such that $C_p^{k_0, h_0}$ tends to infinity if p does so and

$$(5.6) \qquad C_p^{k_0, h_0} / C_p^{k,h} \geq C > 0 \qquad \text{for every } (k, h) \neq (k_0, h_0),$$

where $C_p^{k,h} = \left\| \int_\Gamma \frac{C_{k,h}(\tau_p', \lambda, \sigma_p')}{P(\tau_p', \lambda, \sigma_p')} e^{-is\lambda} d\lambda \right\|_{L^2(\boldsymbol{R}_+^1)} (\boldsymbol{Im}\, \lambda_{k,h}^+(\tau_p', \sigma_p'))^{-\frac{1}{2}} (\boldsymbol{Re}\, \tau_p')^{\nu+1}$.

First we choose $g_p \in C_0(\boldsymbol{R}_+^1)$ which is identically not zero and satisfies

$$
(5.7)\qquad \begin{aligned}
&\left|\int_0^\infty g_p(s)\,ds\int_\Gamma \frac{C_{k,h}(\tau'_p, \lambda, \sigma'_p)}{P(\tau'_p, \lambda, \sigma'_p)}\,e^{-is\lambda}\,d\lambda\right| \\
&\geq \frac{1}{2}\|g_p\|_{L^2(\boldsymbol{R}^1_+)}\left\|\int_\Gamma \frac{C_{k,h}(\tau'_p, \lambda, \sigma'_p)}{P(\tau'_p, \lambda, \sigma'_p)}\,e^{-is\lambda}\,d\lambda\right\|_{L^2(\boldsymbol{R}^1_+)}.
\end{aligned}
$$

Replacing τ', σ' and g in (5. 5) by τ'_p, σ'_p and g_p respectively, we obtain from (5. 7) and the definition of $C_p^{k,h}$

$$
(5.8)\qquad \begin{aligned}
\|\varphi\|_{L^2(\boldsymbol{R}^1_+)} \geq C\Big\{&\frac{1}{2}\,C_p^{k_0,h_0}(\boldsymbol{Im}\,\lambda^+_{k_0,h_0}(\tau'_p, \sigma'_p))^{\frac{1}{2}}\left|\int_0^\infty \gamma_{k_0,h_0}(\tau'_p, x, \sigma'_p)\,\varphi(x)\,dx\right| \\
&-\sum_{(k,h)\neq(k_0,h_0)} C_p^{k,h}(\boldsymbol{Im}\,\lambda^+_{k,h}(\tau'_p, \sigma'_p))^{\frac{1}{2}}\left|\int_0^\infty \gamma_{k,h}(\tau'_p, x, \sigma'_p)\,\varphi(x)\,dx\right|\Big\}.
\end{aligned}
$$

Next if $\boldsymbol{Im}\,\lambda^+_{k_0,h_0}(\tau_0, \sigma'_0)=0$ we take $\varphi_p(x)=\mathbf{exp}\,(\overline{ix\gamma_{k_0,h_0}(\tau'_p, \sigma'_p)})$. Then we have $\|\varphi_p\|_{L^2(\boldsymbol{R}^1_+)}=(2\boldsymbol{Im}\,\lambda^+_{k_0,h_0}(\tau'_p, \sigma'_p))^{-\frac{1}{2}}$ and $\left|\int_0^\infty \gamma_{k_0,h_0}(\tau'_p, x, \sigma'_p)\varphi_p(x)\,dx\right|=(2\,\boldsymbol{Im}\,\lambda^+_{k_0,h_0}(\tau'_p, \sigma'_p))^{-1}$. Moreover the second term in the right hand of (5. 8) is bounded. Hence it follows from them, (5. 6) and (5. 8) that $1\geq C\ C_p^{k_0,h_0}\Big(\frac{1}{2}-C'(\boldsymbol{Im}\,\lambda^+_{k_0,h_0}(\tau'_p, \sigma'_p))\Big)$. By the choice of (k_0, h_0) this inequality is not valid for a sufficiently large p.

If $\boldsymbol{Im}\,\lambda^+_{k_0,h_0}(\tau'_0, \sigma'_0)>0$ we can find $\varphi\in L^2(\boldsymbol{R}^1_+)$ which has a compact support and satisfies $\int_0^\infty \gamma_{k_0,h_0}(\tau'_0, x, \sigma'_0)\,\varphi(x)\,dx=1$ and $\int_0^\infty \gamma_{k,h}(\tau'_0, x, \sigma'_0)\,\varphi(x)\,dx=0$ for every $(k, h)\neq(k_0, h_0)$, because the $\gamma_{k,h}(\tau'_0, x, \sigma_0)$ are linearly independent in $L^2(0, 1)$. By the continuity of the functions $\int_0^\infty \gamma_{k,h}(\tau', x, \sigma')\varphi(x)\,dx$ at (τ'_0, σ'_0), we obtain, for a sufficiently large p,

$$
(5.9)\qquad \begin{aligned}
&\int_0^\infty \gamma_{k_0,h_0}(\tau'_p, x, \sigma'_p)\,\varphi(x)\,dx\geq\frac{1}{2}, \\
&\int_0^\infty \gamma_{k,h}(\tau'_p, x, \sigma'_p)\,\varphi(x)\,dx\leq\varepsilon_p^{k,h} \qquad \text{if } (k, h)\neq(k_0, h_0),
\end{aligned}
$$

where $\varepsilon_p^{k,h}$ is sufficiently small. Therefore it follows from (5. 8) and (5. 9) that

$$
\|\varphi\|_{L^2(\boldsymbol{R}^1_+)}\geq C\ C_p^{k_0,h_0}\Big\{\frac{1}{4}(\boldsymbol{Im}\,\lambda^+_{k_0,h_0}(\tau'_p, \sigma'_p))^{\frac{1}{2}}-\sum_{(k,h)\neq(k_0,h_0)}\varepsilon_p^{k,h}(\boldsymbol{Im}\,\lambda^+_{k,h}(\tau'_p, \sigma'_p))^{\frac{1}{2}}\Big\}.
$$

By the choice of (k_0, h_0) this inequality is not valid for a sufficiently large p. Thus the proof is complete.

6. Applications.

In this section we prove S. Agmon's results in [1] and the interesting results stated in Introduction. First to prove S. Agmon's results we need the following lemma which is implicitly contained in [7].

Lemma 6.1. *Let the condition (♯) be satisfied. Then for every non zero (τ_0, σ_0) with $\boldsymbol{Re}\,\tau_0=0$ and $\sigma_0\in\boldsymbol{R}^{n-1}$ there exists a neighbourhood $U(\tau_0, \sigma_0)$ such that*

1). if a real root in λ of $P(\tau_0, \lambda, \sigma_0)=0$ is simple then there is an analytic function $\lambda(\tau, \sigma)$ in $U(\tau_0, \sigma_0)$ which satisfies $P(\tau, \lambda(\tau, \sigma), \sigma)=0$ and

$$|\boldsymbol{Im}\,\lambda(\tau, \sigma)|\geq C(\boldsymbol{Re}\,\tau) \quad \text{in } U(\tau_0, \sigma_0). \tag{6.1}$$

2). if a real root in λ of $P(\tau_0, \lambda, \sigma_0)=0$ is strictly double then there are analytic functions $\lambda^{\pm}(\tau, \sigma)$ in $U(\tau_0, \sigma_0)\cap\Sigma_+$ which satisfy $P(\tau, \lambda^{\pm}(\tau, \sigma), \sigma)=0$ and

$$|\boldsymbol{Im}\,\lambda^{\pm}(\tau, \sigma)|\geq C(\boldsymbol{Re}\,\tau), \tag{6.2}$$

$$|\boldsymbol{Im}\,\lambda^{+}(\tau, \sigma)\,\boldsymbol{Im}\,\lambda^{-}(\tau, \sigma)|\,|\lambda^{+}(\tau, \sigma)-\lambda^{-}(\tau, \sigma)|^2\geq C(\boldsymbol{Re}\,\tau)^2 \quad \text{in } U(\tau_0, \sigma_0)\cap\Sigma_+. \tag{6.3}$$

Then we obtain the following

Theorem 6.2. (*S. Agmon*) *Suppose that the condition (♯) is satisfied. If $R(\tau, \sigma)\neq 0$ for every non zero $(\tau, \sigma)\in\bar{\boldsymbol{C}}_+\times\boldsymbol{R}^{n-1}$, then the mixed problem (P, B_j) is L^2-well-posed (with decreasing order 0).*

Proof. By the similar consideration in the proof of Theorem 5.1 and using that $R(\tau, \sigma)\neq 0$ for every non zero $(\tau, \sigma)\in\bar{\boldsymbol{C}}_+\times\boldsymbol{R}^{n-1}$, we can show that the mixed problem (P, B_j) is *L^2-well-posed* if and only if the following condition is satisfied:

For every $(\tau_0', \sigma_0')\in(\bar{\Sigma}_+-\Sigma_+)$ there exist a neighbourhood $U(\tau_0', \sigma_0')$ and a constant $C(\tau_0', \sigma_0')$ such that, for any $(\tau', \sigma')\in U(\tau_0', \sigma_0')\cap\Sigma_+$,

$$\begin{aligned}&|C_j(\tau', \lambda_k^-(\tau', \sigma'), \sigma')|^2\\ &\leq C(\tau_0', \sigma_0')|\boldsymbol{Im}\,\lambda_j^+(\tau', \sigma')\,\boldsymbol{Im}\,\lambda_k^-(\tau', \sigma')|\,|(\partial_\lambda P)(\tau', \lambda_k^-(\tau', \sigma'), \sigma')|^2(\boldsymbol{Re}\,\tau')^{-2}\end{aligned} \tag{6.4}$$

where the j and k satisfy the conditions $\boldsymbol{Im}\,\lambda_j^+(\tau_0', \sigma_0')=0$ and $\boldsymbol{Im}\,\lambda_k^-(\tau_0', \sigma_0')=0$ respectively.

First if $\lambda_k^-(\tau_0', \sigma_0')$ is simple then (6.4) is valid by (6.1) and (6.2). When $\lambda_k^-(\tau_0', \sigma_0')$ is strictly double we denote another branch by λ_k^+ (c.f. Lemma 6.1). Next if $\lambda_k^-(\tau_0', \sigma_0')$ is strictly double and $k\neq j$, by the fact that $|C_j(\tau', \lambda_k^-(\tau', \sigma'), \sigma')|\leq C|\lambda_k^+(\tau', \sigma')-\lambda_k^-(\tau', \sigma')|$ and $|(D_\lambda P)(\tau', \lambda_k^-(\tau', \sigma'), \sigma')|\geq C|\lambda_k^+(\tau', \tau')-\lambda_k^-(\tau', \sigma')|$, (6.4) is valid. Finally if $\lambda_k^-(\tau_0', \sigma_0')$ in strictly double and $j=k$ then (6.4) is valid by (6.3). Thus the proof is complete.

Next we prove the following

Theorem 6.3. *Let $Q(D)$ be a homogeneous differential operator, which does not contain the odd order terms in D_x, of order $m-1$ with constant* *coefficients and let* $B_j(D)=D_x^{2j-1}(j=1, \cdots, l; m=2l)$. *If* $P(D)$ *satisfies the condition* (♯) *and does not contain the odd order terms in* D_x, *then the mixed problem* $(P(D)+\varepsilon D_x Q(D), B_j(D))$ *is not* L^2*-well-posed* (*with decreasing order* 0) *for a sufficiently small* ε *with certain fixed sign.*

Proof. We may assume that, for a sufficiently small ε, $L_\varepsilon(D)=P(D)+\varepsilon D_x Q(D)$ is strictly hyperbolic and satisfies the condition (♯). Furthermore the number of the roots $\lambda_j^+(\tau, \sigma)\,(\lambda_j^-(\tau, \sigma))$ of $L_\varepsilon(\tau, \lambda, \sigma)=0$ is a constant l for any $(\tau, \sigma)\in \boldsymbol{C}_+\times \boldsymbol{R}^{n-1}$. Since Lopatinskii's determinant written in the following form:

$$R(\tau, \sigma)=\lambda_1^+(\tau, \sigma)\cdots\lambda_l^+(\tau, \sigma)\prod_{1\le j<k\le l}(\lambda_j^+(\tau, \sigma)+\lambda_k^+(\tau, \sigma)),$$

$S(\tau)$ is empty for any $\tau\in\boldsymbol{C}_+$.

Since $P(D)$ is strictly hyperbolic and $\deg_\tau Q<\deg_\tau P$, there is a point $(\tau_0', \sigma_0')\in(\bar{\Sigma}_+-\Sigma_+)$ such that $P(\tau_0', 0, \sigma_0')=0$ and $Q(\tau_0', 0, \sigma_0')\neq 0$. By the assumption of $P(D)$ we have $(\partial_\lambda L_\varepsilon)(\tau_0', 0, \sigma_0')=\varepsilon Q(\tau_0', 0, \sigma_0')\neq 0$. Hence there exist a neighbourhood $U(\tau_0', \sigma_0')$ and a simple root, denote $\lambda_1^+(\tau', \sigma')$, in $U(\tau_0', \sigma_0')$ such that $\lambda_1^+(\tau_0', \sigma_0')=0$. Here we may assume that $\boldsymbol{Im}\,\lambda_1^+(\tau', \sigma')>0$ in $U(\tau_0', \sigma_0')\cap\Sigma_+$, because one can change ε into $-\varepsilon$.

First we consider the case when there exists a root, denote $\lambda_1^-(\tau', \sigma')$, such that $\lambda_1^-(\tau_0', \sigma_0')\neq 0$ and $\boldsymbol{Im}\,\lambda_1^-(\tau_0', \sigma_0')=0$. Assume that $\boldsymbol{Im}\,\lambda_k^-(\tau_0', \sigma_0')=0\,(k=1, \cdots, h)$. Since $\lambda_k^-(\tau', \sigma')$ $(k=1, \cdots, h)$ are simple in a small neighbourhood $U(\tau_0', \sigma_0')\cap\Sigma_+$, we obtain

$$\begin{aligned}(6.5)\qquad &\int_\Gamma \frac{C_1(\tau', \lambda, \sigma')}{L_\varepsilon(\tau', \lambda, \sigma')}e^{-is\lambda}d\lambda\\ &=\sum_{k=1}^{h}\frac{C_1(\tau', \lambda_k^-(\tau', \sigma'), \sigma')}{(\partial_\lambda L_\varepsilon)(\tau', \lambda_k^-(\tau', \sigma'), \sigma')}e^{-is\lambda_k^-(\tau', \sigma')}+\int_{\Gamma_1}\frac{C_1(\tau', \lambda, \sigma')}{L_\varepsilon(\tau', \lambda, \sigma')}e^{-is\lambda}d\lambda,\end{aligned}$$

where Γ_1 is a closed Jordan curve in the lower half λ-plane enclosing all the roots $\lambda_j^-(\tau', \sigma')\,(j=h+1, \cdots, l)$. Multiplying $\mathbf{exp}\,(\overline{-is\lambda_1^-(\tau', \sigma')})$ to (6.5), it follows from Schwarz inequality that

$$\begin{aligned}(6.6)\qquad &\left\|\int_\Gamma \frac{C_1(\tau' \lambda, \sigma')}{L_\varepsilon(\tau', \lambda, \sigma')}e^{-is\lambda}d\lambda\right\|_{L^2(\boldsymbol{R}_+^1)}(2\,\boldsymbol{Im}\,\lambda_1^-(\tau', \sigma'))^{-\frac{1}{2}}\\ &\ge\left|\sum_{k=1}^{h}\frac{C_1(\tau', \lambda_k^-(\tau, \sigma'), \sigma')}{(\partial_\lambda L_\varepsilon)(\tau', \lambda_k^-(\tau', \sigma'), \sigma')}(\lambda_k^-(\tau', \sigma')-\overline{\lambda_1^-(\tau', \sigma')})^{-1}\right|\\ &\quad-\int_{\Gamma_1}\frac{C_1(\tau', \lambda, \sigma')}{L_\varepsilon(\tau', \lambda, \sigma')}|\lambda-\overline{\lambda_1^-(\tau', \sigma')}|^{-1}d\lambda.\end{aligned}$$

By (5.2) we have

$$(6.7)\qquad C_1(\tau', \lambda, \sigma')=\lambda\prod_{j\neq 1}(\lambda^2-\lambda_j^+(\tau', \sigma')^2)\Big/\Big\{\lambda_1^+(\tau', \sigma')\prod_{j\neq 1}(\lambda_1^+(\tau', \sigma')^2-\lambda_j^+(\tau', \sigma')^2)\Big\}.$$

Then it follows from (6. 6) and (6. 7) that

$$
\begin{aligned}
&\left\|\int_{\Gamma}\frac{C_1(\tau',\lambda,\sigma')}{L_\varepsilon(\tau',\lambda,\sigma')}e^{-is\lambda}d\lambda\right\|_{L^2(\boldsymbol{R}^1_+)} \\
&\geq C|\lambda_1^+(\tau',\sigma')|^{-1}\left\{|\lambda_1^-(\tau',\sigma')||\boldsymbol{Im}\lambda_1^-(\tau',\sigma')|^{-\frac{1}{2}}-C'|\boldsymbol{Im}\lambda_1^-(\tau',\sigma')|^{\frac{1}{2}}\right\}.
\end{aligned}
\tag{6. 8}
$$

If (5.3) in Theorem 5. 1 is valid for $C_1(\tau',\lambda,\sigma')$ then we have from (6. 8)

$$|\boldsymbol{Im}\lambda_1^+(\tau',\sigma')\boldsymbol{Im}\lambda_1^-(\tau',\sigma')|^{\frac{1}{2}}(\boldsymbol{Re}\tau')^{-1}\geq C|\lambda_1^+(\tau',\sigma')|^{-1}(|\lambda_1^-(\tau',\sigma')|-C'|\boldsymbol{Im}\lambda_1^-(\tau',\sigma')|).$$

Using the fact that $|\lambda_1^+(\tau',\sigma')|\leq C(\boldsymbol{Re}\,\tau')$ in a small neighbourhood $U(\tau_0',\sigma_0')$, this inequality is not valid for (τ',σ') sufficiently close to (τ_0',σ_0').

Next we consider the case when $\boldsymbol{Im}\,\lambda_j^-(\tau_0',\sigma_0')<0$ for all $j=1,\cdots,l$. In a sufficiently small neighbourhood $U(\tau_0',\sigma_0')\cap\Sigma_+$, we arrange the roots $\lambda_j^-(\tau',\sigma')$ into r groups $\{\lambda_{k,h}^-(\tau',\sigma')\,;\ h=1,\cdots,k'\}$ $(k=1,\cdots,r)$ such that $\{\lambda_{k,h}^-(\tau_0',\sigma_0')\,;\ h=1,\cdots,k'\}$ is k'-multiple root. By the condition (♯) there is a simple root, denote $\lambda_1^-(\tau',\sigma')$, in a small neighbourhood $U(\tau_0',\sigma_0')\cap\Sigma_+$. Let $\Gamma_k(k=1,\cdots,r)$ be a sufficiently small and closed Jordan curve in the lower half λ-plane enclosing all the roots $\lambda_{k,h}^-(\tau',\sigma')$ $(h=1,\cdots,k')$. Then we can find $g\in L^2(\boldsymbol{R}^1_+)$ such that

$$\left|\int_0^\infty e^{-is\lambda}g(s)\,ds\right|\geq\frac{1}{2}\ \text{ on }\Gamma_1\ \text{ and }\ \left|\int_0^\infty e^{-is\lambda}g(s)\,ds\right|\leq\varepsilon_k\ \text{ on }\Gamma_k\ \text{ for }k\neq 1\,. \tag{6. 9}$$

By Schwarz inquality we obtain

$$
\begin{aligned}
&\|g\|_{L^2(\boldsymbol{R}^1_+)}\left\|\int_{\Gamma}\frac{C_1(\tau',\lambda,\sigma')}{L_\varepsilon(\tau',\lambda,\sigma')}e^{-is\lambda}d\lambda\right\|_{L^2(\boldsymbol{R}^1_+)} \\
&\geq\left|\int_{\Gamma_1}\frac{C_1(\tau',\lambda,\sigma')}{L_\varepsilon(\tau',\lambda,\sigma')}d\lambda\int_0^\infty e^{-is\lambda}g(s)\,ds\right|-\sum_{k\neq 1}\left|\int_{\Gamma_k}\frac{C_1(\tau',\lambda,\sigma')}{L_\varepsilon(\tau',\lambda,\sigma')}d\lambda\int_0^\infty e^{-is\lambda}g(s)\,ds\right|.
\end{aligned}
\tag{6. 10}
$$

If (5. 3) in Theorem 5. 1 is valid for $C_1(\tau',\lambda,\sigma')$ then we have, by (6. 7), (6. 9) and (6. 10),

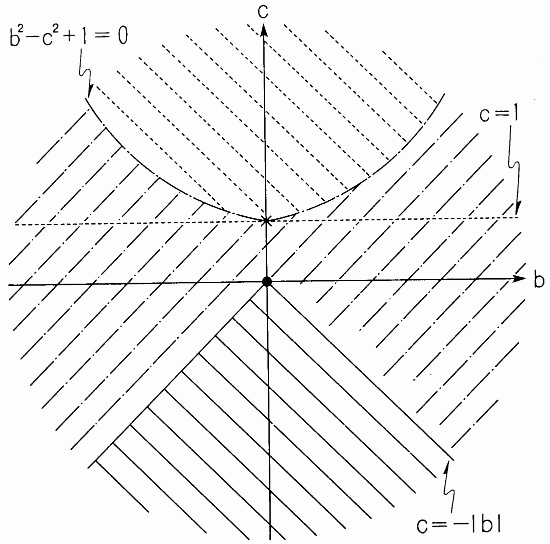
$$\|g\|_{L^2(\boldsymbol{R}^1_+)}(\boldsymbol{Im}\,\lambda_1^+(\tau',\sigma'))^{\frac{1}{2}}(\boldsymbol{Re}\,\tau')^{-1}\geq C|\lambda_1^+(\tau',\sigma')|^{-1}(1-\varepsilon_k)\,.$$

By the fact that $|\lambda_1^+(\tau',\sigma')|\leq C(\boldsymbol{Re}\,\tau')$ in a small neighbourhood $U(\tau_0',\sigma_0')$, this inequality is not valid for (τ',σ') sufficiently close to (τ_0',σ_0'). Thus the proof is complete.

7. Examples.

In this section we present some examples.

1). $P(D)=\partial_t^2-\Delta$ and $B(D)=D_x+bD_y-ic\partial_t$, where Δ is Laplacian in $\boldsymbol{R}^2_+$ and b and c are real. Then we have the following classification:



——, · ; L^2-well-posed (with decreasing order 0),
—·—·— ; not L^2-well-posed but L^2-well-posed with decreasing order 1,
·······, × ; not L^2-well-posed with any decreasing order.

Proof. In this case, $\lambda^{\pm}(\tau, \sigma)=\pm i(\tau^2+\sigma^2)^{\frac{1}{2}}$ where it is assumed that $(\tau^2+\sigma^2)^{\frac{1}{2}}$ has a positive real part if $\tau\in \boldsymbol{C}_+$, Lopatinskii's determinant $R(\tau, \sigma)=i(\tau^2+\sigma^2)^{\frac{1}{2}}+b\sigma-ic\tau$ and reflection coefficient $C(\tau, \lambda^-(\tau, \sigma), \sigma)=(-i(\tau^2+\sigma^2)^{\frac{1}{2}}+b\sigma-ic\tau)/R(\tau, \sigma)$. Hence, it is the necessary and sufficient condition for L^2-well-posedness with decreasing order ν that the following inequality holds in a small neighbourhood in $\Sigma_+\cap V'^c$ of any $(\tau_0', \sigma_0')\in(\bar{\Sigma}_+-\Sigma_+)\cup V'$

$$
(7.1)\qquad \begin{aligned} &|C(\tau', \lambda^-(\tau', \sigma'), \sigma')|^2\\ &\leq C|\boldsymbol{Im}\,\lambda^+(\tau', \sigma')\,\boldsymbol{Im}\,\lambda^-(\tau', \sigma')|\,|(\partial_\lambda P)(\tau', \lambda^-(\tau', \sigma'), \sigma')|^2(\boldsymbol{Re}\,\tau')^{-2\nu-2}. \end{aligned}
$$

Remark that, by Lemma 6.1, the right hand of (7.1) for $\nu=0$ is bounded below.

If $c<-|b|$, $R(\tau, \sigma)\neq 0$ for $(\tau, \sigma)\in \bar{\boldsymbol{C}}_+\times \boldsymbol{R}^1$. By Theorem 6.2, the mixed problem (P, B) is then L^2-well-posed. If $c=-|b|$, $R(\tau, \sigma)\neq 0$ for $(\tau, \sigma)\in \boldsymbol{C}_+\times \boldsymbol{R}^1$ and $R(\tau_0', \sigma_0')=0$ for some $(\tau_0', \sigma_0')\in(\bar{\Sigma}_+-\Sigma_+)$ but $C(\tau', \lambda^-(\tau', \sigma'), \sigma')$ is bounded in a neighbourhood of (τ_0', σ_0'). By the remark above and (7.1) for $\nu=0$, the mixed problem (P, B) is L^2-well-posed. If $b^2-c^2+1<0$ and $c>0$, $S(\tau)$ depends on $\tau\in \boldsymbol{C}_+$. Hence, by Theorem 3.1, the mixed problem (P, B) is not L^2-well-posed with any decreasing order. If $c=1$, $S(\tau)=\{0\}$ but (7.1) for any ν is not valid in a nighbourhood of $(\tau_0', 0)$ $(\tau_0'\in \boldsymbol{C}_+)$. Hence the mixed problem (P, B) is not L^2-well-posed with any decreasing order. In the other case, $R(\tau, \sigma)\neq 0$

for $(\tau, \sigma)\in \boldsymbol{C}_+ \times \boldsymbol{R}^1$, $R(\tau_0', \sigma_0')=0$ for some $(\tau_0', \sigma_0')\in(\bar{\Sigma}_+ - \Sigma_+)$ and the numerator of $C(\tau', \lambda^-(\tau', \sigma'), \sigma')$ is not zero at (τ_0', σ_0'). Hence the mixed problem (P, B) is not L^2-well-posed. But, by the fact that $|R(\tau', \sigma')| \geq C(\boldsymbol{Re}\,\tau')$ in a neighbourhood of (τ_0', σ_0'), the mixed problem (P, B) is L^2-well-posed with decreasing order 1.

2). $P(D)=(\partial_t^2 - a_1\Delta)(\partial_t^2 - a_2\Delta)$ where Δ is Laplacian in $\boldsymbol{R}^n$ and a_1 and a_2 are positive and distinct.

If $B_1(D)=I$ and $B_2(D)=D_x$ then, by Theorem 6.2, the mixed problem (P, B_j) is L^2-well-posed. If $B_1(D)=I$ and $B_2(D)=D_x^2$ or $B_1(D)=D_x$ and $B_2(D)=D_x^3$, by Theorem 5.1 and more precisely taking the residue in (5.3) (c.f. (6.4)), then the mixed problem (P, B_j) is L^2-well-posed. If $B_1(D)=D_x$ and $B_2(D)=D_x^2$ or $B_1(D)=D_x^2$ and $B_2(D)=D_x^3$, then the mixed problem (P, B_j) is not L^2-well-posed but L^2-well-posed with decreasing order 1. Finally if $B_1(D)=I$ and $B_2(D)=D_x^3$, then the mixed problem (P, B_j) is not L^2-well-posed with any decreasing order by Corollary 3. 2.

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